An analysis of the two-vortex case in the Chern-Simons Higgs model*

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June 1997

Abstract

Extending work of Caffarelli-Yang and Tarantello, we present a variational existence proof for two-vortex solutions of the periodic Chern-Simons Higgs model and analyze the asymptotic behavior of these solutions as the parameter coupling the gauge field with the scalar field tends to 0.

1 Introduction

The present note deals with a simplified form of the so-called anyon model, a classical field theory defined on (2+1) dimensional Minkowski space where the Lagrangian couples Maxwell and Chern-Simons terms coming from a gauge field with a scalar field. Multivortex solutions of this model are of interest for example in superconductivity. Since the corresponding Euler-Lagrange equations are somewhat complicated and since at large distances and low energies, the lower order Chern-Simons terms dominates the higher order Maxwell term, Hong-Kim-Pac [HKP] and Jackiw-Weinberg [JW] proposed to study the simplified version without the Maxwell term. Furthermore, they found that for a special 6^{th} order choice of the potential, there may exist time-independent vortex solutions satisfying a first order Bogomolny type

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self-dual equations, similar to the Abelian Higgs equations that arise from a theory with a 4^{th} order potential. Nevertheless, the former theory is mathematically and physically richer, because the potential admits both symmetric and asymmetric minima, and varying the coupling parameter allows to interpolate between the different types of vacua.

Caffarelli-Yang [CY] and Tarantello [T] then obtained existence results for stationary periodic multivortices. Mathematically, this means that the stationary theory is studied on a two-dimensional torus. While the existence results hold for an arbitrary number of prescribed vortices, some of the finer variational aspects and asymptotic results for a coupling parameter tending to 0 could be obtained only for the case of a single vortex. In the present paper, we take up those studies and analyze the two-vortex case. The two-vortex case is more difficult because it turns out to be a borderline case for the Moser-Trudinger inequality. For more than two vortices, special periodic situations may be reduced to the one or two-vortex case, but the general situation is unclear at present.

Acknowledgement:This paper was written at the Max-Planck-Institute for Mathematics in the Sciences in Leipzig. The first author thanks the Max-Planck-Institute for its hospitality and good working conditions. The third author was supported by a fellowship of the Humboldt Foundation. The fourth author was supported by the DFG through the Leibniz award of the second author.

2 Geometrical description of the model

The so-called Chern-Simons Higgs functional introduced by Hong-Kim-Pac [HKP] and Jackiw-Weinberg [JW] has the Lagrangian density

$$\mathcal{L}(\phi, A) = D_\alpha\phi\overline{D_\alpha\phi} + \frac{1}{4}k\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_\gamma - V(|\phi|) \tag{1}$$

for a complex scalar field ϕ coupled with a Chern-Simons gauge field A on 2+1 dimensional Minkowski space $\mathbb{R}^{1,2}$. In geometric terminology, ϕ can be considered as a section of the bundle $\mathbb{R}^{1,2}\times\mathbb{C}$, and $A = -iA_\alpha dx^\alpha$ ($A_\alpha \in \mathbb{R}$, $x = (x^0, x^1, x^2) \in \mathbb{R}^{1,2}$) is a unitary connection on this bundle. (We normalize the electric charge to be 1 here.) The curvature of A is

$$F_A = \frac{-i}{2}F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $\alpha, \beta = 0, 1, 2$, and we write for the covariant derivative

$$D_A\phi = D_\alpha \phi dx^\alpha \qquad \text{with } D_\alpha \phi = \partial\phi - iA_a\phi.$$

In (1) V is a potential term, and in the model under consideration, it is taken as

$$V(|\phi|) = \frac{1}{k^2}|\phi|^2(1 - |\phi|^2)^2. \tag{2}$$

Here, the coupling parameter k is the same as in (1). This choice of V will lead us to Bogomolny type selfduality equations, following the derivation in [JW]. Finally, $\epsilon^{\alpha\beta\gamma}$ is normalized by $\epsilon^{012} = 1$, and of course, the usual summation convention is in force; and we use the Minkowski metric to raise and lower indices. Thus, all notations in (1) have been explained.

The Euler-Lagrange equations for the Lagrangian action density $\mathcal{L}$ are

$$\begin{array}{rcccl} \frac{1}{2}k\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta} & = & j^\gamma & = & i(\phi\overline{D^\gamma\phi} - \overline{\phi}D^\gamma\phi), \\ D_\alpha D^\alpha\phi & = & -\frac{\partial V(\phi)}{\partial\phi}, & & \end{array} \tag{3}$$

where j^γ is the conserved matter current density. [HKP] and [JW] are interested in time independent vortex solutions to these field equations. For such static configuration, the energy becomes

$$E(\phi, A) = \int d^2x(|D_A\phi|^2 - A_0^2|\phi|^2 - kA_0F^{12} + V(|\phi|)) \tag{4}$$

Varying it w.r.t A_0 yields

$$A_0 = -\frac{k}{2}\frac{F^{12}}{|\phi|^2},$$

and E becomes

$$E(\phi, A) = \int d^2x(|D_A\phi|^2 + \frac{k^2}{4}\frac{|F|^2}{|\phi|^2} + V(|\phi|)). \tag{5}$$

With the choice (2) for $V(|\phi|)$, (5) may be rewritten as

$$\begin{array}{rcl} E(\phi, A) & = & \int d^2x(|(D_1 \pm D_2)\phi|^2 + (\frac{k}{|\phi|}F_{12} \mp \frac{2}{k}|\phi|(|\phi|^2 - 1))^2 \\ & & \pm F_{12} + \text{Im }\{\partial_j\varepsilon_{jk}\overline{\phi}D_k\phi\}). \end{array} \tag{6}$$

E is gauge invariant in the sense that its value is unaffected under changing (ϕ, A) to $(e^{i\vartheta}\phi, A_j + \partial_j\vartheta)$ with real valued ϑ.

Following Caffarelli-Yang [CY], we wish to study solutions that are periodic w.r.t. some lattice on $\mathbb{R}^2$. Such solutions can be interpreted as solutions on a fundamental domain

$$\Omega = \{z^1\tau_1 + z^2\tau_2, 0 < z^1, z^2 < 1\}$$

for the lattice, where $\tau_1, \tau_2 \in \mathbb{R}^2$ are the generators of this lattice, satisfying so-called 't Hooft boundary conditions that we now state. Let ϑ_1, ϑ_2 be (smooth) real-valued functions defined for $x = z^2\tau_2$ and $x = \tau_1 + z^2\tau_2$, $0 < z^2 < 1$, or $x = z^1\tau_1$ and $x = \tau_2 + z^1\tau_1$, $0 < z^1 < 1$, respectively. We then require on those boundary lines of Ω where ϑ_j is defined

$$e^{i\vartheta_j(x+\tau_j)}\phi(x + \tau_j) = e^{i\vartheta_j(x)}\phi(x), \tag{7}$$

$$(A_k + \partial_k\vartheta_j)(x + \tau_j) = (A_k + \partial_k\vartheta_j)(x). \tag{8}$$

Since ϕ is a single-valued complex function, its phase change when traversing $\partial\Omega$ has to be $2\pi N$ for some integer N. Using the resulting constraint on the ϑ_j from (7) in (8) yields

$$\int_\Omega F_{12}dx = \int_{\partial\Omega} A_k dx^k = 2\pi N. \tag{9}$$

W.l.o.g., we assume $N \geq 0$ in the sequel, in order to obtain the upper signs in (6) in the sequel. Since the last term in (6) is a divergence term, we obtain

$$\begin{aligned} E(\phi, A) &= \int d^2x \left\{ |(D_1 + iD_2)\phi|^2 + (\frac{k}{2}\frac{F_{12}}{|\phi|} - \frac{1}{k}|\phi|(|\phi|^2 - 1))^2 \right\} \\ &+ 2\pi N \end{aligned} \tag{10}$$

The absolute minima of E therefore satisfy the Bogomolny type self-dual equations

$$D_1\phi + iD_2\phi = 0 \tag{11}$$

$$k^2 F_{12} = 2|\phi|^2(1 - |\phi|^2) \tag{12}$$

subject to the boundary conditions (7) and (8).

Geometrically, we consider the quotient of $\mathbb{R}^2$ by the lattice generated by τ_1, τ_2, i.e. we identify the boundary portion $z^k\tau_k$, $0 \leq z_k \leq 1$,

of Ω with $z^k\tau_k + \tau_j$, for $j \neq k \in \{1, 2\}$. The resulting torus is denoted by Σ. Because of the boundary conditions, ϕ can then be considered as a section of a complex line bundle over Σ of degree N. Equation (11) then says that ϕ is a holomorphic section of this line bundle, and as such, it has to have N zeros, counted with multiplicity (unless $\phi \equiv 0$).

Conversely, it was shown by Caffarelli-Yang [CY] that for some critical number k_c that satisfies the bound

$$0 < k_c \leq \frac{1}{2}\left(\frac{|\Sigma|}{\pi N}\right)^{\frac{1}{2}} \qquad (|\Sigma| := \text{ Area}(\Sigma)) \tag{13}$$

and $0 < k < k_c$, and given $p_1, p_2, \cdots, p_N \in \Sigma$ (not necessarily distinct), there exists such a solution of (11) and (12) with zeros at p_i with corresponding multiplicities in the case some of the p_i coincide. Tarantello [T] then showed that in this situation, there exists a second solution different from the one of [CY]. It is of physical and mathematical interest to study the asymptotic behavior of these solutions as k tends to 0. Tarantello showed that the Caffarelli-Yang solution converges to one in absolute value away from the vortices $p_1, p_2, \cdots, p_N$. In other words, in the limit, we obtain a covariantly constant section of a flat bundle over $\Sigma \setminus \{p_1, p_2, \cdots, p_N\}$, whereas the curvature becomes a sum of delta functions supported on the vortex set. This is analogous to the situation in [HJS] where the Ginzburg-Landau theory with a potential term of fourth order was investigated. In case $N = 1$, Tarantello showed in contrast that the second solution converges to 0 uniformly, and furthermore, after rescaling, one obtains a solution of the mean field equation

$$\begin{cases} \Delta w_0 = -4\pi N \left(\frac{e^{u_0 + w_0}}{\int_\Sigma e^{u_0 + w_0}} - \frac{1}{|\Sigma|} \right) \\ \int_\Sigma w_0 = 0 \end{cases} \tag{14}$$

where $u_0 \in H^{1,p}(\Sigma)$ $(1 < p < 2)$ is the (negative) Green function satisfying

$$\begin{cases} \Delta u_0 = -\frac{4\pi N}{|\Sigma|} + 4\pi \sum_{j=1}^N \delta_{p_j} \\ \int_\Sigma u_0 = 0 \end{cases} \tag{15}$$

In order to see the reason for the limitation $N = 1$ in the last result, we need to describe the method of the proof. The aforementioned existence results were obtained by a sub/supersolution method, i.e. a method based on the maximum principle. In addition, Caffarelli-Yang

introduced a variational method that we are going to describe that could reprove the existence results in case $N = 1$ only.

We can see that (11) is

$$\overline{\partial}\phi - \frac{i}{2}(A_1 + iA_2)\phi = 0 \tag{16}$$

with $\overline{\partial} = 1/2(\partial_1 + i\partial_2)$. Thus, if we know ϕ, A is determined by

$$(A_1 + iA_2) = -2i\overline{\partial}\log\phi. \tag{17}$$

In order to obtain ϕ, we put

$$v(x) = \log|\phi|^2. \tag{18}$$

Because of the gauge invariance, given v, we may obtain ϕ as

$$\phi(x) = \exp(\frac{1}{2}v(x) + i\sum_{j=1}^{N}\arg(x - p_j)). \tag{19}$$

Using (17), (12) translates into the equation for v

$$\Delta v = \frac{4}{k^2}e^v(e^v - 1) + 4\pi\sum_{j=1}^{N}\delta_{p_j}. \tag{20}$$

Writing $v = u_0 + u$ with the Green function u_0 of (15), and putting $K = e^{u_0}$, $\lambda = \frac{4}{k^2}$, we obtain the equation

$$\Delta u = \lambda K e^u(Ke^u - 1) + \frac{4\pi N}{|\Sigma|}. \tag{21}$$

For finding the second solution, Tarantello [T] used the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Sigma(|Du|^2 + \frac{\lambda}{2}(Ke^u - 1)^2) + \frac{4\pi N}{|\Sigma|}\int_\Sigma u. \tag{22}$$

I_λ is well-defined on $H^{1,2}(\Sigma)$ as $K = e^{u_0} \in L^\infty(\Sigma)$ and since the Moser-Trudinger inequality ([M], [A]) holds:

For every $\epsilon > 0$, there exists $C(\epsilon)$ with

$$\log\int_\Sigma e^w \leq (\frac{1}{16\pi} + \epsilon)\int_\Sigma|Du|^2 + \frac{1}{|\Sigma|}\int_\Sigma w + C(\epsilon) \tag{23}$$

for all $w \in H^{1,2}(\Sigma)$.

One easily verifies that critical points of I_λ yield solutions of (21). (The reader may worry about the log on the l.h.s. of (23) when compared with (22), but that is handled by imposing the normalization $\int_\Sigma Ke^u = 1$.)

The reason why the variational method could made to work to obtain a solution of (21) via a minimization procedure in [CY] and to find the asymptotic behavior of the second solution in [T] only in case $N = 1$ stems from the constant $\frac{1}{16\pi}$ in (23)). ($\frac{1}{16\pi}$ in fact is the optimal value in (23), see [DJLW1] and [F]). As one sees from comparing (22) and (23), the case $N = 2$ in that integral leads to a critical or borderline situation for the Moser-Trudinger inequality. Such borderline cases occur in many problems of geometric and analytic interest, see e.g. the monograph [St] of Struwe, and they typically require an analysis that is both subtle and interesting. We therefore attack the variational aspects of the $N = 2$ vortex case in the present paper. We prove the following theorem.

Theorem 2.1 *Let Σ be a flat torus, N=2. The equations (11) and (12) have two solutions (A_k^1, ϕ_k^1) and (A_k^2, ϕ_k^2) for sufficiently small k which satisfy that $|\phi_k^1| \to 1$ almost everywhere as $k \to 0$, and that $|\phi_k^2| \to 0$ almost everywhere as $k \to 0$.*

The first solution resembles the topological solution in the non-compact setting and the second one resembles the non-topological solution. The first solution (A_k^1, ϕ_k^1) was obtained first by Caffarelli-Yang [CY], using a sub/supersolution method. The problem studied in the present paper is meaningful and interesting also on Riemann surfaces other than a torus. Thus, in our companion paper [DJLW2], we study the $N = 2$ case on S^2 where we can find even three solutions with different asymptotic behaviors. In [DJLW1], we derive some background results on the Kazdan-Warner problem on higher genus Riemann surfaces of which the present problem can be considered as a special case.

3 Proof of the theorem

For simplicity we assume that $|\Sigma| = 1$. The following Moser-Trudinger inequality on a Riemann surface was proved in [F] and [DJLW1].

Lemma 3.1 *Let Σ be a compact Riemann surface (with a conformal metric for which $|\Sigma| = 1$). There exists a constant $C > 0$ such that,*

$$\log \int_\Sigma e^u \leq \frac{1}{16\pi} \int_\Sigma |\bigtriangledown u|^2 + \int_\Sigma u + C$$

for any $u \in H^{1,2}(\Sigma)$.

$\square$

We also need the following concentration lemma, which can be seen as a generalization of one on S^2 proved by Chang-Yang [ChY] using the conformal transformations.

Proposition 3.2 *(Concentration lemma) Let M be a compact Riemann surface with volume 1. Given a sequence of functions $u_j \in H^{1,2}(M)$ with $\int_M e^{u_j} = 1$ and $\| \bigtriangledown u_j\|_{L^2(M)} + 16\pi \int_M u_j \leq C$, then either*

(i) *there is a constant $C_0 > 0$ such that $\int_M |\bigtriangledown u_j|^2 \leq C_0$ or*

(ii) *a subsequence which is also denoted by u_j concentrates at a point $p \in M$, i.e., for any $r > 0$,*

$$\lim_{j\to\infty} \int_{B_r(p)} e^{u_j} = 1.$$

Proof: Suppose that (ii) does not hold. That is, every subsequence of u_j does not concentrate. Then for any $p \in M$ there is $0 < r < \frac{1}{16} i_M$ (i_M is the injectivity radius of M) such that

$$\lim_{j\to\infty} \int_{B_r(p)} e^{u_j} < \delta < 1. \tag{24}$$

At least there exists a subsequence u_{j_k} satisfying the above inequality. For simplicity, we shall often implicitly pass to subsequences in the sequel without mentioning it explicitly.

Since M is compact, we can see that there is a finite set $\{ (p_l, r_l) \mid l = 1, 2, \cdots, L \}$ satisfying

$$\lim_{j\to\infty} \int_{B_{r_l}(p_l)} e^{u_j} < \delta < 1.$$

and $\bigcup_{l=1}^L B_{r_l}(p_l) \supset M$.

We assume that

$$\lim_{j\to\infty}\int_{B_{r_1}(p_1)} e^{u_j} = \max\{\lim_{j\to\infty}\int_{B_{r_l}(p_l)} e^{u_j}|l=1,2,\cdots,L\}.$$

So, we have

$$\lim_{j\to\infty}\int_{B_{r_1}(p_1)} e^{u_j} \geq \alpha_0 > 0,$$

for some positive constant α_0.

The improved Moser-Trudinger inequality (see Lemma 3.3 below) implies that

$$\lim_{j\to\infty}\int_{M\setminus B_{2r_1}(p_1)} e^{u_j} = 0,$$

if $\int_M |\bigtriangledown u_j|^2$ is not bounded.

We choose a normal coordinate system (x_1, x_2) around p_1, and we may assume further that in $B_{16r_1}(p_1)$

$$\frac{1}{2}|x-y| \leq dist_M(x,y) \leq 2|x-y|.$$

where $|x-y| = dist_{R^2}(x,y)$.

We consider the square $P_1 = \{|x_i| \leq 4r_1|i=1,2\} \subset R^2$, and we have

$$\lim_{j\to\infty}\int_{\exp(P_1)} e^{u_j} = 1.$$

We divide P_1 into 4^2 equal subsquares. If $\int_M |\bigtriangledown u_j|^2$ is not bounded, using the improved Moser-Trudinger inequality one gets a square P_2 which is a union of at most 3^2 of the equal subsquares, such that

$$\lim_{j\to\infty}\int_{\exp(P_2)} e^{u_j} = 1.$$

Then we can get a sequence of squares $P_n \to p_0 \in M$ such that

$$\lim_{j\to\infty}\int_{B_r(p_0)} e^{u_j} = 1$$

for any $r>0$, which contradicts (24). This means that $\int_M |\bigtriangledown u_j|^2$ is bounded. Therefore (i) holds, this proves the proposition.

$\square$

Lemma 3.3 *([A], [CL]) Let Ω_1 and Ω_2 be two subsets of Σ satisfying* $\mathrm{dist}(\Omega_1, \Omega_2) \geq \epsilon_0 > 0$ *and $\alpha_0 \in (0, 1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \epsilon_0, \alpha_0)$ such that*

$$\int_\Sigma e^u \leq c \exp\{\frac{1}{32\pi - \epsilon} \| \bigtriangledown u\|^2 + \int_\Sigma u\}$$

holds for all $u \in H^{1,2}(\Sigma)$ satisfying

$$\frac{\int_{\Omega_1} e^u}{\int_\Sigma e^u} \geq \alpha_0 \qquad \textit{and} \qquad \frac{\int_{\Omega_2} e^u}{\int_\Sigma e^u} \geq \alpha_0.$$

To prove the theorem, we consider the functionals

$$\begin{aligned} J_\lambda^\pm(w) &= \frac{1}{2}\int_\Sigma |\bigtriangledown w|^2 - 8\pi \log \int_\Sigma K e^w \\ &\quad -8\pi \log(1 \mp \sqrt{1 - B_\lambda(w)}) - \frac{8\pi}{1 \mp \sqrt{1 - B_\lambda(w)}} \end{aligned}$$

in the space

$$\mathcal{A}_\lambda = \{w \in H^{1,2}(\Sigma) | \int_\Sigma w = 0, B_\lambda(w) \leq 1\}$$

where

$$B_\lambda(w) = \frac{32\pi}{\lambda} \frac{\int_\Sigma K^2 e^{2w}}{(\int_\Sigma K e^w)^2}.$$

Lemma 3.4 *Let $\mathcal{A}_\lambda^0$ be the interior of $\mathcal{A}_\lambda$. If $w_\lambda^\pm \in \mathcal{A}_\lambda^0$ is a critical point of $J_\lambda^\pm$, then $u_\lambda^\pm = w_\lambda^\pm - \log(\frac{\lambda}{16\pi}\int_\Sigma K e^{w_\lambda^\pm}) - \log(1 \mp \sqrt{1 - B_\lambda(w_\lambda^\pm)})$ is a solution of (21).*

Proof: If $w_\lambda^\pm \in \mathcal{A}_\lambda^0$ is a critical point of $J_\lambda^\pm$ in $\mathcal{A}_\lambda$, then it satisfies

$$\begin{aligned} \triangle w_\lambda^\pm &= -8\pi(1 + \frac{B_\lambda(w_\lambda^\pm)}{1 \mp \sqrt{1 - B_\lambda(w_\lambda^\pm)}}) \frac{K e^{w_\lambda^\pm}}{\int_\Sigma K e^{w_\lambda^\pm}} \\ &\quad + 8\pi \frac{B_\lambda(w_\lambda^\pm)}{1 \mp \sqrt{1 - B_\lambda(w_\lambda^\pm)}} \frac{K^2 e^{2w_\lambda^\pm}}{\int_\Sigma K^2 e^{2w_\lambda^\pm}} + \mu \end{aligned}$$

Integrating yields that $\mu = 8\pi$. Computing directly shows that $u_\lambda^\pm$ is a solution of (21). This proves the lemma.

$\square$

Note that, for any fixed $\lambda > 0$, we have $B_\lambda(w) \geq \frac{32\pi}{\lambda}$ by the Hölder inequality. We can see that $J_\lambda^\pm$ are bounded from below in $\mathcal{A}_\lambda$. Therefore we can get two sequences of functions $w_i^\pm \in \mathcal{A}_\lambda$ such that

$$J_\lambda^\pm(w_i^\pm) \to \inf_{w \in \mathcal{A}_\lambda} J_\lambda^\pm(w)$$

as $i \to \infty$.

We claim that $w_i^\pm$ are bounded in $H^{1,2}(\Sigma)$. We show this claim for w_i^+, the proof for w_i^- is similar. It suffices to show that $\| \bigtriangledown w_i^+\|_{L^2(\Sigma)}$ is bounded.

We set $u_i^+ = w_i^+ + c_i^+$, where c_i^+ is a constant such that $\int_\Sigma e^{u_i^+} = 1$. The boundedness of $J_\lambda^+(w_i^+)$ implies

$$\frac{1}{2}\int_\Sigma |\bigtriangledown u_i^+|^2 + 8\pi c_i^+ \leq C \tag{25}$$

for some constant C. By the definition of c_i^+, the previous inequality implies that u_i^+ satisfies the conditions in Proposition 3.2. On the other hand, since $w_i^+ \in \mathcal{A}_\lambda$, we have

$$\frac{32\pi}{\lambda}\frac{\int_\Sigma K^2 e^{2u_i^+}}{(\int_\Sigma K e^{u_i^+})^2} \leq 1.$$

So

$$\int_\Sigma K^2 e^{2u_\lambda^{+i}} \leq C_\lambda$$

where C_λ is a positive constant depending only on λ. By the Hölder inequality one gets

$$\int_{B_\epsilon(p)} K e^{u_i^+} \leq C_\lambda \epsilon.$$

Now applying Proposition 3.2 to the sequence u_i^+, we get the boundedness of $\| \bigtriangledown u_i^+\|_{L^2(\Sigma)}$, which is equivalent to our claim.

From the claim, we have

(1) u_i^+ converges to u_λ^+ weakly in $H^{1,2}(\Sigma)$,

(2) u_i^+ converges to u_λ^+ strongly in $L^q(\Sigma)$, for any $1 < q < \infty$,

(3) $\int_\Sigma (Ke^{u_i^+})^l \to \int_\Sigma (Ke^{u_\lambda^+})^l$ as $i \to \infty$, for $l = 1, 2$, by the Moser-Trudinger inequality and the Lebesgue convergence theorem.

(2) and (3) imply that $u_\lambda^+ \in \mathcal{A}_\lambda$. The semi-continuity of the Dirichlet integral, together with (1)–(3), implies that

$$J_\lambda^+(u_\lambda^+) = \inf_{u \in \mathcal{A}_\lambda} J_\lambda^+(u).$$

Therefore, to show the theorem, it suffices to prove that $J_\lambda^\pm$ achieves its minimum in $\mathcal{A}_\lambda^0$ if λ is sufficiently large.

By Lemma 3.1 we have

$$\inf_{w \in \partial \mathcal{A}_\lambda} J_\lambda^+(w) \geq -C - 8\pi(1 + \log \max K), \tag{26}$$

however, choosing $w_0 = 0$, we have

$$\begin{aligned}\inf_{w \in \mathcal{A}_\lambda} J_\lambda^+(w) \ \ &\leq \ \ -8\pi \log \int_\Sigma K - 8\pi \log(1 - \sqrt{1 - B_\lambda(0)}) \\ &\quad -\frac{8\pi}{1 - \sqrt{1 - B_\lambda(0)}}.\end{aligned} \tag{27}$$

By (26) and (27) we can see that J_λ^+ achieves its minimum in $\mathcal{A}_\lambda^0$ if λ is sufficiently large.

For J_λ^-, with the help of Lemma 3.1 we can show that

$$\inf_{w \in \partial \mathcal{A}_\lambda} J_\lambda^-(w) \geq \alpha_0 - 8\pi + 8\pi \log(16\pi) \tag{28}$$

and

$$\lim_{\lambda \to \infty} \inf_{w \in \mathcal{A}_\lambda} J_\lambda^-(w) = \alpha_0 - 4\pi + 8\pi \log(8\pi), \tag{29}$$

where $\alpha_0 = \inf_{u \in H^{1,2}(\Sigma)} \{\frac{1}{2} \int_\Sigma |\bigtriangledown u|^2 + 8\pi \int_\Sigma u - 8\pi \log \int_\Sigma e^u\}$. The argument for (28) and (29) is the same as one given in [DJLW2]. We omit it here. Since $-8\pi + 8\pi \log(16\pi) > -4\pi + 8\pi \log(8\pi)$, (28) and (29) imply that J_λ^+ achieves its minimum in $\mathcal{A}_\lambda^0$ provided that λ is sufficiently large.

By Lemma 3.3, we therefore have two solutions of the equations (11) and (12), (A_k^1, ϕ_k^1) and (A_k^2, ϕ_k^2) corresponding to $u_\lambda^\pm$. It was proved in [T] that $|\phi_k^1| \to 1$ almost everywhere as $k \to 0$.

It is clear that

$$|\phi_k^2|^2 = e^{u_0 + u_\lambda^-},$$

so

$$\int_\Sigma |\phi_k^2|^2 = \frac{16\pi}{\lambda}\frac{1}{1+\sqrt{1-B_\lambda(w_\lambda^-)}} \leq \frac{16\pi}{\lambda}.$$

Thus, as $\lambda = \frac{4}{k^2}$,

$$\lim_{k\to 0}\int_\Sigma |\phi_k^2|^2 = 0.$$

This proves our theorem.

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